The Hölder continuity of a class of 3-dimension ultraparabolic equations

WANG Wendong and ZHANG Liqun *

Institute of Mathematics, AMSS, Academia Sinica, Beijing

Abstract

We obtained the C^α continuity for weak solutions of a class of ultraparabolic equations with measurable coefficients of the form

$$\partial_t u = \partial_x(a(x,y,t)\partial_x u) + b_0(x,y,t)\partial_x u + b(x,y,t)\partial_y u,$$

which generalized our recent results on KFP equations.

keywords: Hypoelliptic, ultraparabolic equations, Hölder regularity

1 Introduction

Consider a class of ultraparabolic operator on R^{2+1}:

$$Lu \equiv \partial_x(a(x,y,t)\partial_x u) + b_0(x,y,t)\partial_x u + b(x,y,t)\partial_y u - \partial_t u = 0, \tag{1.1}$$

where $(x,y,t) = z \in \Omega \subset R^{2+1}$, $a(z)$, $b_0(z)$ and $b(z)$ is real, measurable functions. We assume that $b(z)$ is twice differentiable, and there exists a positive constant μ such that for $z \in \Omega$,

$$\mu < a(z) < \mu^{-1}, \qquad \frac{\partial b(z)}{\partial x} \neq 0, \qquad |b|_{C^2} + |b_0|_\infty \leq \mu^{-1}. \tag{1.2}$$

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Also, we denote

$$L_0 u = \partial_x^2 u + x\partial_y u - \partial_t u = 0, \tag{1.3}$$

$$L_1 u = \partial_x(a(x,y,t)\partial_x u) + x\partial_y u - \partial_t u = 0, \tag{1.4}$$

and

$$L_2 u = \partial_x(a(x,y,t)\partial_x u) + b_0(x,y,t)\partial_x u + x\partial_y u - \partial_t u = 0. \tag{1.5}$$

We remark that the equation (1.3) and (1.4) are the examples of 3-dimension homogeneous Kolmogorov-Fokker-Planck equations (or KFP equations). The condition $\frac{\partial b}{\partial x} \neq 0$ ensures (1.1) satisfies Hörmander's hypoellipticity conditions,

$$\text{rank Lie}(\partial_x, b\partial_y - \partial_t)(z) = 3, \quad \forall z \in \Omega.$$

The study of regularity of the KFP equation has a long history, and the earlier works are mainly on the Schauder type estimates. The study of regularity of weak solutions is begun in recent years. A recent paper of Pascucci and Polidoro [6], has proved that the Moser iterative method still works for the class of KFP equations with measurable coefficients. By the same technique, Cinti, Pascucci, Polidoro [1] consider a class of nonhomogeneous KFP equations, and Cinti, Polidoro [2] deal with a more general ultraparabolic equation. Their results show that for a non-negative sub-solution u of the ultraparabolic equation, L^∞ norm of u is bounded by the L^p norm ($p \geq 1$). The second author [10], [11] has proved C^α property of weak solutions by Kruzhkov's approach for homogeneous KFP equations, and the authors deal with nonhomogeneous KFP equations in [7]. By simplifying the cut-off function and generalizing their earlier arguments, the authors [8] have considered more general ultraparabolic equations whose fundamental solution is implicit.

We are not try to review the detailed history, but focus on the study of the Hölder continuity of a simple looking case. In this paper, we give another generalization of KFP equations in R^{2+1} and consider the hypoelliptic operator as L in (1.1).

We say that u is a *weak solution* if it satisfies (1.1) in the distribution sense, that is for any $\phi \in C_0^\infty(\Omega)$, Ω is a open subset of R^{2+1}, then

$$\int_\Omega \phi(b_0\partial_x + b\partial_y - \partial_t)u - a\partial_x\phi\partial_x u = 0, \tag{1.6}$$

and u, $\partial_x u$, $b\partial_y - \partial_t u \in L^2_{\text{loc}}(\Omega)$.

Our main result is the following theorem:

Theorem 1.1 *Under the assumption (1.2), the weak solution of (1.1) is Hölder continuous.*

2 Some Preliminary and Known Results

We follow the earlier notations to give some basic known properties related to our problems. For more details of the subject, we refer to Pascucci and Polidoro [6] and Lanconelli and Polidoro [5].

Let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $E(\tau) = \exp(-\tau \mathrm{B}^{\mathrm{T}}) = \begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix}$.

For $(x, y, t), (\xi, \eta, \tau) \in R^{2+1}$, set

$$(x, y, t) \circ (\xi, \eta, \tau) = ((\xi, \eta) + E(\tau)\tbinom{x}{y}, t + \tau),$$

then $(R^{2+1}, \circ)$ is a Lie group with identity element $(0, 0)$, and the inverse of an element is $(x, y, t)^{-1} = (-E(-t)\tbinom{x}{y}, -t)$. The left translation by (ξ, η, τ) given by

$$(x, y, t) \mapsto (\xi, \eta, \tau) \circ (x, y, t),$$

is a invariant translation to the operator L_0. The associated dilation to operator L_0 is given by

$$\delta_t = diag(t, t^3, t^2),$$

where t is a positive parameter, and the homogeneous dimension of $(R^{2+1}, \circ)$ with respect to the dilation δ_t is 6.

The norm in R^{2+1}, related to the group of translations and dilation to the equation is defined by $||(x, y, t)|| = r$, if r is the unique positive solution to the equation

$$\frac{x^2}{r^2} + \frac{y^2}{r^6} + \frac{t^2}{r^4} = 1,$$

where $(x, y, t) \in R^{2+1} \setminus \{0\}$. And $||(0, 0)|| = 0$. Obviously

$$\|\delta_\mu(x, y, t)\| = \mu\|(x, y, t)\|,$$

for all $(x, y, t) \in R^{2+1}$.

The ball centered at a point (x_0, t_0) is defined by

$$\mathcal{B}_r(x_0, t_0) = \{(x, t)| \quad ||(x_0, t_0)^{-1} \circ (x, t)|| \leq r\},$$

and

$$\mathcal{B}_r^-(x_0, t_0) = \mathcal{B}_r(x_0, t_0) \cap \{t < t_0\}.$$

For convenience, we sometimes use the cube instead of the balls. The cube at point $(0, 0)$ is given by

$$\mathcal{C}_r(0, 0) = \{(x, y, t)| \quad |x| \leq r, |y| \leq 8r^3, \quad |t| \leq r^2\}.$$

It is easy to see that there exists a constant Λ such that

$$\mathcal{C}_{\frac{r}{\Lambda}}(0, 0) \subset \mathcal{B}_r(0, 0) \subset \mathcal{C}_{\Lambda r}(0, 0).$$

For L_0, the fundamental solution $\Gamma(\cdot, \zeta)$ with pole in $\zeta = (\xi, \eta, \tau) \in R^{2+1}$ is smooth except at the diagonal of $R^{2+1} \times R^{2+1}$. It has the following form

at $\zeta = 0$,

$$\Gamma(z) = \Gamma(z,0) = \begin{cases} \frac{\sqrt{3}}{2\pi t^2}\exp[-\frac{1}{t}(x^2 + \frac{3}{t}xy + \frac{3}{t^2}y^2)] & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leq 0; \end{cases} \tag{2.1}$$

and

$$\Gamma(z,\zeta) = \begin{cases} \frac{\sqrt{3}}{2\pi(t-\tau)^2}\exp[-\frac{x^2+x\xi+\xi^2}{t-\tau} - \frac{3(x+\xi)(y-\eta)}{(t-\tau)^2} - \frac{3(y-\eta)^2}{(t-\tau)^3}] & \text{if} \quad t > \tau, \\ 0 & \text{if} \quad t \leq \tau. \end{cases} \tag{2.2}$$

Obviously we can derive from the above formula,

$$\int_{R^2} \Gamma(x,y,t;\xi,\eta,\tau)dxdy = \int_{R^2} \Gamma(x,y,t;\xi,\eta,\tau)d\xi d\eta = 1, \quad \text{if} \quad t > \tau, \tag{2.3}$$

and

$$\Gamma(\delta_\mu \circ z) = \mu^{-4}\Gamma(z), \quad \forall z \neq 0,\ \mu > 0. \tag{2.4}$$

A *weak sub-solution* of (1.5) in a domain Ω is a function u such that u, $\partial_x u$, $(x\partial_y - \partial_t)u \in L^2_{loc}(\Omega)$ and for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$,

$$\int_\Omega \phi(b_0\partial_x + x\partial_y - \partial_t)u - (\partial_x u)a\partial_x\phi \geq 0. \tag{2.5}$$

We recall a result of Pascucci and Polidoro obtained by using the Moser's iterative method (see [6]) states as following

Lemma 2.1 *Let u be a non-negative weak sub-solution of (1.4) in Ω. Let $(x_0,t_0) \in \Omega$ and $\overline{\mathcal{B}_r^-(x_0,t_0)} \subset \Omega$ and $p \geq 1$. Then there exists a positive constant C which depends only on the operator L such that, for $0 < r \leq 1$*

$$\sup_{\mathcal{B}^-_{\frac{r}{2}}(x_0,t_0)} u^p \leq \frac{C}{r^6}\int_{\mathcal{B}_r^-(x_0,t_0)} u^p, \tag{2.6}$$

provided that the last integral converges.

The second author [10] proved the following result.

Theorem 2.1 *If u is a weak solution of (1.4), then u is Hölder continuous.*

Using the same technique, we can obtain the similar result to the equation (1.5).

Theorem 2.2 *If u is a weak solution of (1.5), then u is Hölder continuous.*

In section 3, we shall sketch the proof of this theorem first. We mainly focus on the proof of the oscillation estimates. Then we give a transformation as Weber in [9] and complete the proof of Theorem 1.1.

We make use of a classical potential estimates (see (1.11) in [3]) here to prove the Poincaré type inequality.

Lemma 2.2 *Let $(R^{N+1}, \circ)$ is a homogeneous Lie group of homogeneous dimension $Q+2$, $\alpha \in (0, Q+2)$ and $G \in C(R^{N+1} \setminus \{0\})$ be a δ_μ-homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(R^{N+1})$ for some $p \in (1, \infty)$, then*

$$G_f(z) \equiv \int_{R^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $C = C(Q, p)$ such that

$$||G_f||_{L^q(R^{N+1})} \leq C \max_{||z||=1} |G(z)| \quad ||f||_{L^p(R^{N+1})}, \tag{2.7}$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}.$$

Corollary 2.1 *Let $f \in L^2(R^{2+1})$, and recall the definitions in [6]*

$$\Gamma(f)(z) = \int_{R^{2+1}} \Gamma(z, \zeta) f(\zeta) d\zeta, \qquad \forall z \in R^{2+1},$$

and

$$\Gamma(\partial_\xi f)(z) = -\int_{R^{2+1}} \partial_\xi \Gamma(z,\zeta) f(\zeta) d\zeta, \qquad \forall z \in R^{2+1},$$

then exists an absolute constant C *such that*

$$\|\Gamma(f)\|_{L^{2\tilde{k}}(R^{2+1})} \le C\|f\|_{L^2(R^{2+1})}, \tag{2.8}$$

and

$$\|\Gamma(\partial_\xi f)\|_{L^{2k}(R^{2+1})} \le C\|f\|_{L^2(R^{2+1})}, \tag{2.9}$$

where $\tilde{k} = 3$, $k = \frac{3}{2}$.

3 Proof of Theorem 2.2

Outline of the proof of Theorem 2.2:

Step 1: L^∞ estimate via Moser iteration. It can be checked that the same Caccioppoli type inequality holds (See Theorem 3.1, [6]), since

$$\int_{B_1} \psi^2 b_0 \partial_x (v^2) \le \frac{1}{2} \int_{B_1} \psi^2 a v^2 + C(\mu) \int_{B_1} \psi^2 v^2.$$

In order to use the Moser iteration, one need to prove a Sobolev type inequality. It can be proved that the Sobolev type inequality holds for non-negative weak sub-solution (See Theorem 3.3 in [6]). Here one may deal with $\int_{B_r} [\Gamma(z,\cdot) v b_0 \partial_x \psi](\zeta)$ as I_2 in [6]. Then one can obtain the L^∞ estimate as in Lemma 2.1.

Step 2: Oscillation estimates.

This is obtained in Lemma 3.6. We shall focus on this parts in the following discussions.

Step 3: Hölder regularity.

This is followed by the oscillation estimated by a standard argument.

Now we turn to the proof of main results. We may consider the local estimate at a ball centered at $(0,0)$, since the equation (1.4) is invariant under the left translation when a is constant. We follow the same route as [10], [7] and [8]. For convenience, we consider the estimates in the following cube, instead of $\mathcal{B}_r^-$,

$$\mathcal{C}_r^- = \{(x,y,t)| \quad -r^2 \le t < 0,\, |x| \le r,\, |y| \le (2r)^3\}.$$

Let

$$K_r = \{(x,y)|\, |x| \le r,\, |y| \le (2r)^3\}.$$

Let $0 < \alpha, \beta < 1$ be constants, for fixed t and h, we denote

$$\mathcal{N}_{t,h} = \{(x,y)|\, (x,y) \in K_{\beta r},\, u(x,y,t) \ge h\}.$$

We sometimes abuse the notations of $\mathcal{B}_r^-$ and $\mathcal{C}_r^-$, since there are equivalent.

Lemma 3.1 *Suppose that $u(x,t) \ge 0$ be a solution of equation (1.5) in $\mathcal{C}_r^-$ centered at $(0,0)$ and*

$$mes\{(x,t) \in \mathcal{C}_r^-, \quad u \ge 1\} \ge \frac{1}{2} mes(\mathcal{C}_r^-),$$

then there exist constants α, β and h_1, $0 < \alpha, \beta, h_1 < 1$, where h_1 only depends on μ, such that for almost all $t \in (-\alpha r^2, 0)$ and $0 < h < h_1$

$$mes\{\mathcal{N}_{t,h}\} \ge \frac{1}{11} mes\{K_{\beta r}\}.$$

Proof: Let

$$v = \ln^+\left(\frac{1}{u + h^{\frac{9}{8}}}\right),$$

where h is a constant, $0 < h < 1$, to be determined later. Then v at points where v is positive, satisfies

$$\partial_x(a(x,y,t)\partial_x v) - a(\partial_x v)^2 + b_0(x,y,t)\partial_x v + x\partial_y v - \partial_t v = 0. \tag{3.1}$$

Let $\eta(s)$ be a smooth cut-off function so that

$$\eta(s)=1, \quad \text{for} \quad s<\beta r,$$

$$\eta(s)=0, \quad \text{for} \quad s\geq r.$$

Moreover, $0\leq\eta\leq 1$ and $|\eta'|\leq\frac{2}{(1-\beta)r}$.

Multiplying $\eta(|x|)^2$ to (3.1) and integrating by parts on $K_r\times(\tau,t)$

$$\begin{aligned}
&\textstyle\int_{K_{\beta r}} v(x,y,t)dxdy+\int_\tau^t\int_{K_r}\eta^2\,a|\partial_x v|^2dxdyds\\
\leq\;&\textstyle\int_\tau^t\int_{K_r}\eta^2(\partial_x(a(x,y,t)\partial_x v)+b_0(x,y,t)\partial_x v+x\partial_y v)dxdyds\\
&\textstyle+\int_{K_r} v(x,y,\tau)dxdy\\
\leq\;&\textstyle\frac{C(\mu)}{\beta^4(1-\beta)^2}|K_{\beta r}|+\int_\tau^t\int_{K_r}(\frac{1}{2}\eta^2\,a|\partial_x v|^2+x\partial_y v\eta^2)dxdyds\\
&\textstyle+\int_{K_r} v(x,y,\tau)dxdy, \qquad a.e. \quad \tau,t\in(-r^2,0).
\end{aligned}\tag{3.2}$$

Then

$$\begin{aligned}
|\textstyle\int_{K_r} x\partial_y v\eta^2| &= |\textstyle\int_{|x|\leq r} xv\eta^2\,|_{y=-8\beta^3r^3}^{8\beta^3r^3}\,dx|\\
&\leq \tfrac{1}{4}r^{-2}\beta^{-4}|K_{\beta r}|\ln(h^{-\frac{9}{8}}).
\end{aligned}$$

Integrating by t to I_B, we have

$$|\textstyle\int_\tau^t\int_{K_r} x\partial_y v\eta^2|\leq\frac{1}{4}\beta^{-4}|K_{\beta r}|\ln(h^{-\frac{9}{8}}).\tag{3.3}$$

We shall estimate the measure of the set $\mathcal{N}_{t,h}$. Let

$$\nu(t)=mes\{(x,y)|\quad (x,y)\in K_r,\,u(x,y,t)\geq 1\}.$$

By our assumption, for $0<\alpha<\frac{1}{2}$

$$\frac{1}{2}r^2mes(K_r)\leq\int_{-r^2}^0\nu(t)dt=\int_{-r^2}^{-\alpha r^2}\nu(t)dt+\int_{-\alpha r^2}^0\nu(t)dt,$$

that is

$$\int_{-r^2}^{-\alpha r^2} \nu(t)dt \geq (\frac{1}{2} - \alpha) r^2 mes(K_r),$$

then there exists a $\tau \in (-r^2, -\alpha r^2)$, such that

$$\mu(\tau) \geq (\frac{1}{2} - \alpha)(1-\alpha)^{-1} mes(K_r).$$

By noticing $v = 0$ when $u \geq 1$, we have

$$\int_{K_r} v(x,y,\tau)dxdy \leq \frac{1}{2}(1-\alpha)^{-1} mes(K_r) \ln(h^{-\frac{9}{8}}). \tag{3.4}$$

Now we choose α (near zero), and β (near one), such that

$$\frac{1}{4\beta^4} + \frac{1}{2\beta^4(1-\alpha)} \leq \frac{4}{5}, \tag{3.5}$$

and fix them from now on.

By (3.2), (3.3), (3.4) and (3.5), we deduce

$$\begin{aligned} &\textstyle\int_{K_{\beta r}} v(x,y,t)dxdy \\ &\leq [\tfrac{C(\mu)}{\beta^4(1-\beta)^2} + \tfrac{4}{5}\ln(h^{-\frac{9}{8}})] mes(K_{\beta r}). \end{aligned} \tag{3.6}$$

When $(x,y) \notin \mathcal{N}_{t,h},$, we have

$$\ln(\frac{1}{2h}) \leq \ln^+(\frac{1}{h + h^{\frac{9}{8}}}) \leq v,$$

then

$$\ln(\frac{1}{2h}) mes(K_{\beta r} \setminus \mathcal{N}_{t,h}) \leq \int_{K_{\beta r}} v(x,y,t)dxdy.$$

Since

$$\frac{C + \frac{4}{5}\ln(h^{-\frac{9}{8}})}{\ln(h^{-1})} \longrightarrow \frac{9}{10}, \qquad \text{as} \quad h \to 0,$$

then there exists constant h_1 such that for $0 < h < h_1$ and $t \in (-\alpha r^2, 0)$

$$mes(K_{\beta r} \setminus \mathcal{N}_{t,h}) \leq \frac{10}{11} mes(K_{\beta r}).$$

Then we proved our lemma.

Let $\chi(s)$ be a C^∞ smooth function given by

$$\begin{array}{lll} \chi(s) = 1 & if & s \le \theta^{\frac{1}{6}} r, \\ \chi(s) = 0 & if & s > r, \end{array}$$

where $\theta > 0$ is a constant, to be determined in Lemma 3.4, and $\theta^{\frac{1}{6}} < \frac{1}{2}$. Moreover, we assume that

$$0 \le -\chi'(s) \le \frac{2}{(1-\theta^{\frac{1}{6}})r}, \quad |\chi''(s)| \le \frac{C}{r^2},$$

and for any β_1, β_2, with $\theta^{\frac{1}{6}} < \beta_1 < \beta_2 < 1$, we have

$$|\chi'(s)| \ge C(\beta_1, \beta_2) r^{-1} > 0,$$

if $\beta_1 r \le s \le \beta_2 r$.

For $(x, y) \in R^2$, $t \le 0$, we set

$$\mathcal{Q} = \{(x, y, t) | -r^2 \le t \le 0,\, |x| \le \frac{r}{\theta},\, |y| \le \frac{r^3}{\theta}\}.$$

We define the cut off functions by

$$\phi_0(x, y, t) = \chi([\theta^2 y^2 - 6tr^4]^{\frac{1}{6}}),$$

$$\phi_1(x, y, t) = \chi(\theta|x|),$$

$$\phi(x, y, t) = \phi_0 \phi_1. \tag{3.7}$$

Lemma 3.2 *By the definition of ϕ and the above arguments, we have*

$$(x\partial_y - \partial_t)\phi_0(z) \le 0, \quad \text{for} \quad \text{z} \in \mathcal{Q}. \tag{3.8}$$

And since $\theta^{\frac{1}{6}} < \frac{1}{2}$, we have
(1) $\phi(z) \equiv 1$, in $\mathcal{B}^-_{\theta r}$,

(2) $\mathrm{supp}\phi\bigcap\{(\mathrm{x,y,t})|(\mathrm{x,y})\in \mathrm{R}^2, \mathrm{t}\le 0\}\subset \mathcal{Q}$*,*
(3) there exists a constant α_1*,* $0<\alpha_1<\min\{\alpha, \frac{1}{12}\}$*, such that*

$$\{(x,y,t)|-\alpha_1 r^2\le t<0, (x,y)\in K_{\beta r}\}\subseteq \mathrm{supp}\phi,$$

moreover, $0<\phi_0(z)<1$*, for* $z\in\{(x,y,t)|-\alpha_1 r^2\le t\le -\theta r^2, (x,y)\in K_{\beta r}\}$*.*

Proof: By the definition of ϕ_0, we attain

$$(x\partial_y-\partial_t)\phi_0 \;\; = \chi'([\theta^2y^2-6tr^4]^{\frac{1}{6}})\frac{1}{6}[\theta^2y^2-6tr^4]^{-\frac{5}{6}}[6r^4+2\theta^2xy]\le 0.$$

When $\theta<\frac{1}{6}$, we can check that obviously (1) holds. We notice that either $|x|\ge\frac{r}{\theta}$, or $|y|\ge\frac{r^3}{\theta}$, or $t\le -r^2$, then ϕ vanishes, hence we obtain (2). When $(x,y)\in K_{\beta r}$, then $\phi_1>0$ and we can choose $\theta<\frac{1}{64}$ and t small, for example, $t>-\alpha_1 r^2$, such that $\theta^2y^2-6tr^4<r^6$, then we obtain (3).

Now we have the following Poincaré's type inequality.

Lemma 3.3 *Let* w *be a non-negative weak sub-solution of (1.5) in* $\mathcal{B}_1^-$*. Then there exists an absolute constant* C*, such that for* $r<\theta<1$

$$\int_{\mathcal{B}^-_{\theta r}}(w(z)-I_0)_+^2\le C\theta^2r^2\int_{\mathcal{B}^-_{\frac{r}{\theta}}}|\partial_x w|^2, \tag{3.9}$$

where $I_0=sup_{\mathcal{B}^-_{\theta r}}I_1(z)$*, and*

$$I_1(z)=\int_{\mathcal{B}^-_{\frac{r}{\theta}}}[-\Gamma(z,\zeta)w(\zeta)(\xi\partial_\eta-\partial_\tau)\phi(\zeta)-\partial_\xi^2\phi(\zeta)\Gamma(z,\zeta)w(\zeta)]d\zeta, \tag{3.10}$$

where Γ *is the fundamental solution of* L_0*, and* ϕ *is given by (3.7).*

Proof: We represent w in terms of the fundamental solution of Γ, i.e.

$$\varphi(z)=-\int_{R^{2+1}}\Gamma(z,\zeta)L_0\varphi(\zeta)d\zeta, \quad \forall\varphi\in C_0^\infty(R^{2+1}).$$

By an approximation of ϕ and integrating by parts, for $z \in \mathcal{B}^-_{\theta r}$, we have

$$
\begin{aligned}
w(z) &= \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle \partial_\xi (w\phi)(\zeta), \partial_\xi \Gamma(z,\zeta)\rangle - \Gamma(z,\zeta)(\xi\partial_\eta - \partial_\tau)(w\phi)(\zeta)]d\zeta \\
&= I_1(z) + I_2(z) + I_3(z),
\end{aligned} \tag{3.11}
$$

where

$$
I_1(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\Gamma(z,\zeta)w(\xi\partial_\eta - \partial_\tau)\phi + \langle \partial_\xi\phi, \partial_\xi\Gamma(z,\zeta)\rangle w + \Gamma(z,\zeta)\langle \partial_\xi w, \partial_\xi \phi\rangle]d\zeta,
$$

$$
\begin{aligned}
I_2(z) \quad &= \quad \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle (1-a)\partial_\xi w, \partial_\xi \Gamma(z,\zeta)\rangle\phi - \Gamma(z,\zeta)\langle (a+1)\partial_\xi w, \partial_\xi\phi\rangle \\
& \qquad + \Gamma(z,\zeta)\phi b_0 \partial_\xi w]d\zeta = I_{21} + I_{22} + I_{23},
\end{aligned}
$$

and

$$
I_3(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle a\partial_\xi w, \partial_\xi(\Gamma(z,\zeta)\phi)\rangle - \Gamma(z,\zeta)\phi(b_0\partial_\xi + \xi\partial_\eta - \partial_\tau)w]d\zeta.
$$

Note that $\mathrm{supp}\phi \bigcap \{\tau \le 0\} \subset \mathcal{Q} \subset \overline{\mathcal{B}^-_{\frac{r}{\theta}}}$, $z \in \mathcal{B}^-_{\theta r}$ and $\langle \partial_\xi\phi, \partial_\xi\Gamma(z,\zeta)\rangle$ vanishes in a small neighborhood of z. Integrating by parts we obtain $I_1(z)$ as in (3.10).

From our assumption, w is a weak sub-solution of (1.5), and ϕ is a test function of this semi-cylinder. In fact, we let

$$
\tilde{\chi}(\tau) = \begin{cases} 1 & \tau \le 0, \\ 1 - n\tau & 0 \le \tau \le 1/n, \\ 0 & \tau \ge 1/n. \end{cases}
$$

Then $\tilde{\chi}(\tau)\phi\Gamma(z,\zeta)$ can be a test function (see [6]). As $n \to \infty$, we obtain $\phi\Gamma(z,\zeta)$ as a legitimate test function, and $I_3(z) \le 0$. Then in $\mathcal{B}^-_{\theta r}$,

$$
0 \le (w(z) - I_0)_+ \le I_2(z).
$$

By Corollary 2.1 we have

$$||I_{21}||_{L^2(\mathcal{B}^-_{\theta r})} \leq C\theta r||I_{21}||_{L^3(\mathcal{B}^-_{\theta r})} \leq C\theta r||\partial_\xi w||_{L^2(\mathcal{B}^-_{\frac{r}{\theta}})}. \tag{3.12}$$

Similarly

$$||I_{2i}||_{L^2(\mathcal{B}^-_{\theta r})} \leq C\theta^2 r||\partial_\xi w||_{L^2(\mathcal{B}^-_{\frac{r}{\theta}})}, \quad i = 2, 3$$

then we proved our lemma.

Now we apply Lemma 3.3 to the function $w = \ln^+ \frac{h}{u+h^{\frac{9}{8}}}$. If u is a weak solution of (1.5), then w is a weak sub-solution. We estimate the value of I_0.

Lemma 3.4 *Under the assumptions of Lemma 3.3, there exist constants θ, λ_0, $\lambda_0 < 1$ only depends on constants α and β, such that for $r < \theta$*

$$|I_0| \leq \lambda_0 \ln(h^{-\frac{1}{8}}). \tag{3.13}$$

Proof: We first come to estimate the second term of $I_1(z)$ and as before, denote $z = (x, y, t)$ and $\zeta = (\xi, \eta, \tau)$. Note $z \in \mathcal{B}^-_{\theta r}$, we have

$$\begin{aligned} & \textstyle\int_{\mathcal{B}^-_{\frac{r}{\theta}}} [|\partial_\xi^2 \phi(\zeta)|\Gamma(z,\zeta)w]d\zeta \\ \leq \;& r^2 sup_{\xi\in \mathrm{supp}(\partial_\xi \phi)} |\partial^2_\xi \phi|(\zeta) \ln(h^{-\frac{1}{8}}). \quad (By\ \ (2.3)) \end{aligned}$$

We only need to estimate $|\partial^2_\xi \phi_1|$. Since

$$|\partial_\xi \phi_1| = |\theta\chi'(\theta|\xi|)\partial_\xi|\xi|| \leq 4\theta r^{-1}, \quad |\partial^2_\xi \phi_1| \leq C\theta^2 r^{-2}.$$

Hence

$$|\int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\partial^2_\xi \phi \Gamma(z,\zeta)w]d\zeta| \leq C_3 \theta^2 \ln(h^{-\frac{1}{8}})$$

where C_3 is an absolute constant.

Now we let $w \equiv 1$, then for $z \in \mathcal{B}^-_{\theta r}$, (3.11) gives

$$1 = \textstyle\int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\phi_1 \Gamma(z,\zeta)(\xi\partial_\eta - \partial_\tau)\phi_0]d\zeta + \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\partial^2_\xi \phi(\zeta)\Gamma(z,\zeta)]d\zeta. \tag{3.14}$$

By (3.8) in Lemma 3.2, we know that

$$-\phi_1\Gamma(z,\zeta)(\xi\partial_\eta - \partial_\tau)\phi_0 \geq 0.$$

We only need to prove $-\phi_1\Gamma(z,\zeta)(\xi\partial_\eta - \partial_\tau)\phi_0$ has a positive lower bound in a domain which w vanishes, and this bound independent of r and small θ. So we can find a λ_0, $0 < \lambda_0 < 1$, such that this lemma holds.

For $z \in B^-_{\theta r}$, set

(3.15)
$$\zeta \in Z = \{(\xi,\eta,\tau) | -\alpha_1 r^2 \leq \tau \leq -\frac{\alpha_1}{2}r^2,\ (\xi,\eta) \in K_{\beta r},\ w(\xi,\eta,\tau) = 0\},$$

then by Lemma 3.1, $|Z| = C(\alpha_1,\beta)r^6$.

We note that when $\zeta = (\xi,\eta,\tau) \in Z$ and $\theta < \frac{1}{64}$, $w(\zeta) = 0$, $\phi_1(\zeta) = 1$, and

$$|\chi'([\theta^2 y^2 - 6tr^4]^{\frac{1}{6}})| \geq C(\alpha_1)r^{-1} > 0.$$

Consequently

$$\begin{aligned}
&\textstyle\int_Z [-\phi_1\Gamma(z,\zeta)(\xi\partial_\eta - \partial_\tau)\phi_0]\, d\zeta \\
&= -\textstyle\int_Z \phi_1\Gamma(z,\zeta)\chi'([\theta^2\eta^2 - 6\tau r^4]^{\frac{1}{6}})\frac{1}{6}[\theta^2\eta^2 - 6\tau r^4]^{-\frac{5}{6}}[6r^4 + 2\theta^2\xi\eta]d\zeta \\
&\geq C(\alpha_1)\textstyle\int_Z r^{-2}\Gamma(\zeta^{-1}\circ z, 0)d\zeta = C(\alpha,\beta) = C_4 > 0,
\end{aligned}$$

where we have used $\Gamma(z,\zeta) \geq Cr^{-4}$, as $\tau \leq -\frac{\alpha_1}{2}r^2$ and $z \in B^-_{\theta r}$. In fact, by (2.2) one can obtain this result easily.

By (3.14) and

$$I_0(z) = \sup_{\mathcal{B}^-_{\theta r}} \int_{\mathcal{B}^-_{\frac{r}{\theta}} \setminus Z} [-\Gamma(z,\zeta)w(\zeta)(\xi\partial_\eta - \partial_\tau)\phi(\zeta) - \partial^2_\xi\phi(\zeta)\Gamma(z,\zeta)w(\zeta)]d\zeta,$$

we have

$$|I_0| \leq (1 - C_4 + C_3\theta^2)\ln(h^{-\frac{1}{8}}) + C_3\theta^2\ln(h^{-\frac{1}{8}}). \tag{3.16}$$

We can choose a small θ which is fixed from now on, such that $|I_0| \leq \lambda_0 \ln(h^{-\frac{1}{8}})$, where $0 < r < \theta$, and $0 < \lambda_0 < 1$ which only depends on α and β.

The following two Lemmas are similar to those in [8], we give them for completeness.

Lemma 3.5 *Suppose that $u \geq 0$ is a solution of equation (1.5) in $\mathcal{B}_r^-$ centered at $(0,0)$ and $mes\{(x,y,t) \in \mathcal{B}_r^-,\ u \geq 1\} \geq \frac{1}{2} mes(\mathcal{B}_r^-)$. Then there exist constants θ and h_0, $0 < \theta, h_0 < 1$ which only depend on λ_0 and μ, such that*

$$u(x,y,t) \geq h_0 \quad in \quad \mathcal{B}_{\theta r}^-. \tag{3.17}$$

Proof: We consider

$$w = \ln^+\left(\frac{h}{u + h^{\frac{9}{8}}}\right),$$

for $0 < h < 1$, to be decided. By applying Lemma 3.3 to w, we have

$$\fint_{\mathcal{B}_{\theta r}^-} (w - I_0)_+^2 \leq C \frac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} \int_{\mathcal{B}_r^-} |\partial_x w|^2.$$

Let $\tilde{u} = \frac{u}{h}$, then $\tilde{u}$ satisfies the conditions of Lemma 3.1. We can get similar estimates as (3.2), (3.3), (3.4) and (3.5), hence we have

$$\begin{aligned} & C \tfrac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} \textstyle\int_{\mathcal{B}_r^-} |\partial_x w|^2 \\ & \leq C(\mu) \tfrac{\theta r^2}{|\mathcal{B}_{\theta r}^-|} [\tfrac{C(\mu)}{\beta^4 (1-\beta)^2} + \tfrac{4}{5} \ln(h^{-\frac{1}{8}})] mes(K_{\beta^{-1} r}) \\ & \leq C(\theta, \mu, \beta) \ln(h^{-\frac{1}{8}}), \end{aligned} \tag{3.18}$$

where θ has been chosen. By L^∞ estimate, there exists a constant, still denoted by θ, such that for $z \in \mathcal{B}_{\theta r}^-$,

$$w - I_0 \leq C(\mu, \beta) (\ln(h^{-\frac{1}{8}}))^{\frac{1}{2}}. \tag{3.19}$$

Therefore we may choose h_0 small enough, so that

$$C(\mu,\beta)(\ln(\frac{1}{h_0^{\frac{1}{8}}}))^{\frac{1}{2}} \leq \ln(\frac{1}{2h_0^{\frac{1}{8}}}) - \lambda_0 \ln(\frac{1}{h_0^{\frac{1}{8}}}).$$

Then (3.13) and (3.19) derive

$$\max_{\mathcal{B}^-_{\theta r}} \frac{h_0}{u + h_0^{\frac{9}{8}}} \leq \frac{1}{2h_0^{\frac{1}{8}}},$$

which implies $\min_{\mathcal{B}^-_{\theta r}} u \geq h_0^{\frac{9}{8}}$, then we finished the proof of this Lemma.

Lemma 3.6 *Suppose that u is a weak solution of equation (1.5) in* $\mathcal{B}^-_r$*, then exists constant* h_2*,* $0 < h_2 < 1$*, such that*

$$Osc_{\mathcal{B}^-_{\theta r}} u \leq h_2\, Osc_{\mathcal{B}^-_r} u, \tag{3.20}$$

where θ *is given in Lemma 3.5.*

Proof: We may assume that $M = \max_{\mathcal{B}^-_r}(+u) = \max_{\mathcal{B}^-_r}(-u)$, otherwise we replace u by $u - c$, since u is bounded locally. Then either $1 + \frac{u}{M}$ or $1 - \frac{u}{M}$ satisfies the assumption of Lemma 3.5, and we suppose $1 + \frac{u}{M}$ does, thus Lemma 3.5 implies that there exists $h_0 > 0$ such that

$$\inf_{\mathcal{B}^-_{\theta r}}(1 + \frac{u}{M}) \geq h_0,$$

that is $u \geq M(h_0 - 1)$, then

$$Osc_{\mathcal{B}^-_{\theta r}} u \leq M - M(h_0 - 1) \leq (1 - \frac{h_0}{2}) Osc_{\mathcal{B}^-_r} u,$$

where we can let $h_2 = (1 - \frac{h_0}{2})$.

Proof of Theorem 2.2. By the standard regularity arguments, for example, see Chapter 8 in [4], we can obtain the result near point $(0, 0)$. By the left invariant translation group action, we know that u is C^α in the interior.

4 Proof of Main Theorem

By Theorem 2.2 and a variable transformation (see [9]), we can prove Theorem 1.1.

Proof: Since $\partial_x b(x,y,t) \neq 0$, let $b = \xi$, $y = \eta$, and $t = \tau$, then

$$\frac{\partial}{\partial x} = \frac{\partial b}{\partial x}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} + \frac{\partial b}{\partial y}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial b}{\partial t}\frac{\partial}{\partial \xi},$$

and the equations (1.1) can be written as

$$\frac{\partial b}{\partial x}\frac{\partial}{\partial \xi}(a\frac{\partial b}{\partial x}\frac{\partial}{\partial \xi}\, u) + (b_0\frac{\partial b}{\partial x} + \xi\frac{\partial b}{\partial y} - \frac{\partial b}{\partial t})\frac{\partial}{\partial \xi}u + \xi\partial_\eta u - \partial_\tau\, u = 0. \tag{4.1}$$

From the implicit function theorem, we know $(x,y,t) :\longrightarrow (\xi,\eta,\tau)$ is a C^2 diffeomorphism, and $\frac{\partial b}{\partial x} = (\frac{\partial x(\xi,\eta,\tau)}{\partial \xi})^{-1}$, hence the above equation can attain

(4.2)
$$\frac{\partial}{\partial \xi}(\frac{\partial b}{\partial x}a\frac{\partial b}{\partial x}\frac{\partial}{\partial \xi}\, u) + [a\frac{\partial^2 x}{\partial \xi^2}(\frac{\partial b}{\partial x})^3 + b_0\frac{\partial b}{\partial x} + \xi\frac{\partial b}{\partial y} - \frac{\partial b}{\partial t}]\frac{\partial}{\partial \xi}u + \xi\partial_\eta u - \partial_\tau\, u = 0,$$

that is

$$\frac{\partial}{\partial \xi}(\tilde{a}\frac{\partial}{\partial \xi}\, u) + \tilde{b_0}\frac{\partial}{\partial \xi}u + \xi\partial_\eta u - \partial_\tau\, u = 0, \tag{4.3}$$

where $\tilde{a} = \frac{\partial b}{\partial x}a\frac{\partial b}{\partial x}$, and $\tilde{b_0} = a\frac{\partial^2 x}{\partial \xi^2}(\frac{\partial b}{\partial x})^3 + b_0\frac{\partial b}{\partial x} + \xi\frac{\partial b}{\partial y} - \frac{\partial b}{\partial t}$.

In a fixed bounded domain, $\tilde{a} \in L^\infty$ and $\tilde{b_0} \in L^\infty$, by Theorem 2.2 the weak solution of (4.3) is Hölder continuous.

We give an immediate corollary. Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$ and $m \geq n$.

(4.4)
$$Lu \equiv \sum_{i,j=1}^{m} \partial_{x_i}(a_{ij}(x,y,t)\partial_{x_j}\, u) + \sum_{k=1}^{m} b_0^k(x,y,t)\partial_{x_k}u + \sum_{l=1}^{n} b_l(x,y,t)\partial_{y_l}u - \partial_t\, u = 0,$$

and we assume:

[H.1] the coefficients a_{ij}, $1 \leq i, j \leq m$, are real valued, measurable functions of (x, t). Moreover, $a_{ij} = a_{ji} \in L^\infty(R^{m+n+1})$ and there exists a $\mu > 0$ such that

$$\mu \sum_{i=1}^{m} \xi_i^2 \leq \sum_{i,j=1}^{m} a_{ij}(x, y, t)\xi_i\xi_j \leq \frac{1}{\mu} \sum_{i=1}^{m} \xi_i^2$$

for every $(x, y, t) \in R^{m+n+1}$, and $\xi \in R^m$.

[H.2] $b_0^j \in L^\infty(\Omega)$, $b_l \in C^2(\Omega)$, and $|b_0^j|_\infty, |b_l|_{C^2} \leq \frac{1}{\mu}$, where $j = 1, \ldots, m$ and $l = 1, \ldots, n$. There exists $i_1, \cdots, i_n$, such that $\frac{\partial(b_1,\ldots,b_n)}{\partial(x_{i_1},\ldots,x_{i_n})} \neq 0$, where $i_j \in \{1, \ldots, m\}$, $j = 1, \ldots, n$ and $i_1 < \cdots < i_n$.

Corollary 4.1 *Under the assumption [H.1] and [H.2], the weak solutions of (4.4) are Hölder continuous.*

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